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FREE GROUPOIDS, TREES, AND FREE GROUPS*

J. DUSKIN
Department of Mathematics, State University of New York at Buffalo, Amherst, NY 14260, USA

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To Bernhard Banaschewski on his 60th birthday

Introduction

If $G:\mathbb{G}\to\mathbb{F}$ is a fully faithful functor of groupoids which is, up to isomorphism, surjective on objects, i.e. an *essential equivalence*, then using the axiom of choice, it is well known that one can find a quasi-inverse $G:\mathbb{F}\to\mathbb{G}$ for F which makes it into an equivalence of categories, i.e. so that $FG \stackrel{\sim}{\to} \mathrm{id}(F)$ and $GF \stackrel{\sim}{\to} \mathrm{id}(G)$ and, indeed, this assertion is equivalent to the axiom of choice.[1] What we intend to show here is a direct proof of slightly less obvious result: If $\mathbb{F}$ is a free groupoid on a directed graph $\mathscr{F} \rightrightarrows \mathrm{Ob}(\mathbb{F})$, then it is possible to choose a quasi-inverse for F in such a way that $\mathbb{G}$ is seen to be free on the directed graph $\mathscr{I} \rightrightarrows \mathrm{Ob}(\mathbb{G})$ which consists of the non-identity images under G of the generators of $\mathbb{F}$ together with the natural isomorphisms $\beta_X : GF(X) \stackrel{\sim}{\to} X$, $X\in \mathrm{Ob}(\mathbb{G})$ of the equivalence. An immediate corollary of this theorem is the precise form of the Nielsen–Schreier theorem [4,5] which gives a system of free generators for any subgroup of a free group. Another is Serre's theorem [6,7] which characterizes free groups as those which can be made to operate freely on a graph which is a tree. Both are seen to follow from the essential part of the theorem, Corollary 2.3: *If* $\mathbb{G} \xrightarrow{F} \mathbb{F}$ *is an essential equivalence of a group* $\mathbb{G}$ *with a free groupoid* $\mathbb{F}$ *(so that* $\mathbb{G}$ *is isomorphic to the group of automorphisms of some object of a connected non-empty free groupoid), then* $\mathbb{G}$ *is a free group.* For readers familiar with graph theory, it is followed by an independent proof of this corollary which leads directly to the Nielsen–Schreier theorem (Theorem 4.1). The more general form of the Bass–Serre theorem is amenable to related methods and will be treated in a separate paper.

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[1] For any surjection f, the canonical functor from the graph of the equivalence relation associated with f considered as a groupoid into the discrete groupoid defined by its target is an essential equivalence which admits a quasi-inverse if and only if f admits a section.

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1. Directed graphs and free groupoids

1.0. Throughout this paper the term *graph* $\mathscr{G}: E \rightrightarrows V$ will mean directed graph, i.e. a set E of *edges* and a set V of *vertices* together with functions $S, T: E \rightrightarrows V$, called *source* and *target*, respectively. The notation "$f: X_0 \to X_1$" will simply mean that $f \in E$ with $S(f) = X_0$ and $T(f) = X_1$. An edge f will be called a *loop* if $S(f) = T(f)$. The *opposite graph* $\mathscr{G}^{\mathrm{op}}$ has the same edges and vertices as $\mathscr{G}$ but has its source and target interchanged. It will often be useful to require that the graph has *degeneracies*, i.e. an additional function $I: V \to E$ such that $SI = TI = \mathrm{id}(V)$. For each vertex X_0, the distinguished loop $I(X_0): X_0 \to X_0$ will be called the *degenerate loop* associated with the vertex X_0 and an edge will be said to *degenerate* provided it is of the form $I(X)$ for some vertex X of the graph. The graph will be said to have *formal inverses* if an idempotent bijection $(-)^{-1}: E \to E$ is given which is *direction reversing*, i.e. $S(f^{-1}) = T(f)$ and $T(f^{-1}) = S(f)$ for all $f \in E$, and carries degenerate edges into themselves if they are present. Clearly one can freely adjoin to the set of edges of a graph a copy of the set of vertices of the graph in such a fashion that they become the set of degenerate edges of a new graph with degeneracies. Moreover, one can freely adjoin to the set of non-degenerate edges $E - I(V)$ a copy of itself which will become the set of formal inverses of a graph with formal inverses. In this fashion one can functorially adjoin or forget this additional structure as needed. The most convenient way to do this is via the notion of a path:

1.1. In any graph $\mathscr{G}: E \rightrightarrows V$ a (*coherent*) *path of length* $n \geq 0$ *from a vertex* X_0 *to a vertex* X_n is a sequence $(f_i)_{1 \leq i \leq n}$ of edges of $\mathscr{G}$ for which $X_0 = S(f_1)$, $X_n = T(f_n)$, and $T(f_i) = S(f_{i+1})$ for all i. A path from X_0 to itself is called a *circuit about* X_0, and if of length 0 (the empty sequence of edges), the *degenerate loop about* X_0. The set of paths of length 0 is in bijective correspondence with the set of vertices of the graph, those of length 1 with the edges. The adjunction of degeneracies is now accomplished by the graph which has the same vertices as the original graph $\mathscr{G}$, but whose set of edges is just the set of paths of $\mathscr{G}$ of length ≤ 1. The graph which has the same vertices as the original graph $\mathscr{G}$, but whose set of edges is just the set of paths of $\mathscr{G}$ of length ≤ 1 has an obvious category structure where composition is given by concatenation of paths and identities by the paths of length 0. It is called the *category of paths of the graph* $\mathscr{G}$. Formal inverses may be obtained with the graph $\mathscr{G}[E^{-1}]$ which has the same vertices as $\mathscr{G}$ but whose set of edges is the disjoint union of the edges of $\mathscr{G}$ with those of $\mathscr{G}^{\mathrm{op}}$, $f: X_1 \to X_0 \in \mathscr{G}^{\mathrm{op}}$ becomes $f^{-1}: X_1 \to X_0$ in $\mathscr{G}[E^{-1}]$. The adjunction of both degeneracies and formal inverses becomes just the graph of paths of length ≤ 1 in $\mathscr{G}[E^{-1}]$. The familiar notion of a *chaotic path* from X_0 to X_n in a graph $\mathscr{G}$, i.e. a sequence of vertices and edges linking the vertices together without regard to coherence of source and target, is most easily defined, quite simply, as a path in $\mathscr{G}[E^{-1}]$. In any graph with formal inverses, a subsequence (f_1, f_{i+1}) of a path in which $f_i = f_{i+1}^{-1}$ or $f_i^{-1} = f_{i+1}$ is called a *backtracking*. A path which has no backtracking is said to be a *reduced path*.

1.2. For any graph $\mathscr{G} : E \rightrightarrows V$, we define the *free* (or *path*) *groupoid on* (or *generated by*) *the graph* $\mathscr{G}$ as the groupoid $\mathbb{F}(\mathscr{G})$ which is universal for source and target preserving maps of $\mathscr{G}$ into the underlying graph of a groupoid. In terms of paths, it may be described as the groupoid whose objects are the vertices of $\mathscr{G}$ and whose arrows are the equivalence classes of paths of length ≥ 0 of the graph $\mathscr{G}[E^{-1}]$ in which any path with a backtracking is equivalent to one in which the backtracking is eliminated. A circuit about X_0 which is equivalent to the empty loop $I(X_0)$ is called a *degenerate circuit*. Composition in $\mathbb{F}(\mathscr{G})$ is still given through concatenation of paths which remains associative after reduction. The inverse of a reduced path is the reduced path in which each f_i is replaced by f_i^{-1} in reverse order. The degenerate loops remain the identities with proofs of these assertions parallel to those for free groups. Indeed, if the set of vertices of the graph is reduced to a one element set, the free groupoid on such a *reduced graph* is isomorphic to the free group generated by the set of edges of $\mathscr{G}$.

As with the free groupoid, the adjunction of degeneracies and/or formal inverses provide the corresponding left adjoints to the obvious forgetful functors of these categories (whose morphisms are required to preserve whatever structure is present). Composition of the appropriate functors provides the corresponding left adjoints into the category of groupoids. In any case this amounts to no more than taking the category of paths of the appropriate graph and imposing the relations which make the degeneracies and/or formal inverses into true identities and/or inverses. Thus for example the free groupoid on a graph with degeneracies is isomorphic to the free groupoid on the directed graph $(E - I(V)) \rightrightarrows V$ whose set of edges is the complement of the set of degernate edges in E. Note also that the familiar operation of 'collapsing an edge to a point' is meaningful only in the category of graphs with degeneracies where it may be interpreted as making an edge degenerate (i.e. forming the appropriate pushout over the canonical morphism to the terminal object, which has exactly one vertex and one (degenerate) edge).

$$
\begin{tikzcd}
 & \text{Groupoids} \arrow[d, shift left] & \\
 & \begin{array}{c}\text{Directed Graphs with}\\ \text{Degeneracies and Formal Inverses}\end{array} \arrow[u, shift left] \arrow[dl, shift left] \arrow[dr, shift left] & \\
\begin{array}{c}\text{Directed Graphs with}\\ \text{Degeneracies}\end{array} \arrow[ur, shift left] \arrow[dr, shift left] & & \begin{array}{c}\text{Directed Graphs with}\\ \text{Formal Inverses}\end{array} \arrow[ul, shift left] \arrow[dl, shift left] \\
 & \text{Directed Graphs} \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}
\tag{1}
$$

We now are able to recall the following observations originally due to Higgins [3] (whose philosophy of the usefulness of the category of groupoids in the study of the category of groups we again affirm here). In our formulation they become tautologies:

1.3. Proposition (Higgins [3]). (a) *A (non-empty) graph $\mathscr{G}$ is* connected (*i.e. any pair of vertices may be joined by a (possibly) chaotic path*) *if and only if the free groupoid $\mathbb{F}(\mathscr{G})$ is* connected (*i.e. any two objects are isomorphic*).

(b) *A graph $\mathscr{G}$ is* circuit free (*i.e. there exist no non-degenerate (possibly) chaotic paths from any vertex to itself*) *if and only if $\mathbb{F}(\mathscr{G})$ is* simply connected (*i.e.* Aut(X) *is a trivial group for all $X \in \mathrm{Ob}(\mathbb{F}(\mathscr{G})) = V$.*

(c) *A graph $\mathscr{G}$ is a* tree (*i.e. is connected and circuit free*) *if and only if $\mathbb{F}(\mathscr{G})$ is connected and simply connected (i.e. is isomorphic to the codiscrete groupoid*[2]

$$V \times V \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} V$$

defined by its set of objects).

1.4. Since the filtered union of subtrees of a graph is a tree, by Zorn's Lemma every graph contains a maximal tree which must have the same set of vertices as the graph whenever the graph is connected (otherwise maximality would be contradicted). Consequently, every connected graph has a maximal tree as a wide subgraph. By passage to the free groupoids generated by the tree and the graph we obtain a *functorial* section

$$\begin{tikzcd} \mathbb{F}(i) : \mathbb{F}(\mathscr{T}) \arrow[r, hook] \arrow[d, dash, "\wr"] & \mathbb{F}(\mathscr{G}) \\ V \times V & \end{tikzcd} \tag{2}$$

of the canonical functor $\langle T, S\rangle : \mathbb{F}(\mathscr{G}) \to V \times V$ whose target is the codiscrete groupoid defined by the set of vertices of the graph and whose defining property makes the square

$$\begin{tikzcd} \mathscr{T} \arrow[r, hook] \arrow[d, hook] & \mathscr{G} \arrow[d, hook] \\ V \times V \cong F(\mathscr{T}) \arrow[r, hook] & \mathbb{F}(\mathscr{G}) \end{tikzcd} \tag{3}$$

[2] Higgins [3] calls such groupoids *simplicial*; Gabriel–Zismann [4] call them *simply connected groupoids*. We prefer the term codiscrete since they define the right adjoint to the forgetful functor Ob : **Gpd** → **SET**, whose left adjoint has long been called the *discrete groupoid functor* (i.e. no arrows other than identities).

a pullback (on the underlying graph level). Every maximal tree can be obtained from such a section since any groupoid always contains a maximal codiscrete subgroupoid which must be isomorphic to $\mathrm{Ob}(\mathbb{F})\times\mathrm{Ob}(\mathbb{F})$ if $\mathbb{F}$ is connected.

1.5. On the graph-theoretic level and, indeed for our purposes here, the significance of the existence of a maximal tree in a connected graph is that for any two vertices of the graph there is a *unique* chaotic path which connects them and lies entirely in the tree.

2. Equivalences with free groupoids

We now recall the following definition and proceed with a statement of the main theorem:

Definition 2.0. A functor $F:\mathbb{G}\to\mathbb{F}$ is said to be an *essential equivalence* provided it is *fully faithful* (i.e. $\mathrm{Hom}_{\mathbb{G}}(X,Y)\xrightarrow{F}\mathrm{Hom}_{\mathbb{F}}(F(X),F(Y))$ is a bijection for all objects X,Y in $\mathbb{G}$) and *essentially surjective on objects* (i.e. given any object A in $\mathbb{F}$ there exists an object Z in $\mathbb{G}$ together with an isomorphism $\varphi:F(Z)\xrightarrow{\sim}A$).

A *quasi-inverse* for F is a functor $G:\mathbb{F}\to\mathbb{G}$ such that $GF\xrightarrow{\sim}\mathrm{id}\,\mathbb{F}$ and $FG\xrightarrow{\sim}\mathrm{id}(\mathbb{G})$. The pair F,G is then said to be an *equivalence.*

Granted the axiom of choice, such a G exists for any essential equivalence F, but we can say more, as follows:

Theorem 2.1. *If $F:\mathbb{G}\to\mathbb{F}$ is an essential equivalence of groupoids in which F is free on a graph $\mathscr{I}:E\rightrightarrows V$, then F admits a quasi-inverse G which has the following properties*:

(a) *G is constant on connected components and*

(b) *$\mathbb{G}$ is free on the graph which has the objects of $\mathbb{G}$ for vertices and for edges the set of non-identity images under G of the generators of $\mathbb{G}$ together with the isomorphisms $\beta_X:GF(X)\xrightarrow{\sim}X$, $X\in\mathrm{Ob}(\mathbb{G})$ of the equivalence.*

2.2. Proof. Since any groupoid is the coproduct (= disjoint union) of its *connected components* (i.e., maximal connected subgroupoids) any essential equivalence of groupoids is the coproduct of essential equivalences of connected groupoids. Moreover, since any free groupoid is the coproduct of the free groupoids generated by the connected components of the generating graph, we may restrict our attention to connected groupoids. We will now give two proofs of the theorem, the first of which is 'algebraic' (and imitates Schreier) and the second of which is categorical and is most easily compared with [3] and [4].

Proof 1. Since $\mathbb{F}$ and $\mathbb{G}$ are non-empty we may choose a fixed object Z in $\mathbb{G}$, and a maximal tree $\mathcal{T} \subseteq \mathcal{G}$. We now define the object mapping of G to be $G(A) = Z$ for all objects A in F and define $\alpha_A : F(G(A)) = F(Z) \xrightarrow{\sim} A$ to be that unique path from $F(Z)$ to A in $\mathbb{F}(\mathcal{G})$ which is defined by the corresponding unique chaotic path in the tree $\mathcal{T}$. For any generating edge $f : A \to B$, $G(f) : Z \to Z$ is defined as that unique arrow in $\mathbb{G}$ whose image under F is equal to $\alpha_B^{-1} f \alpha_A : F(Z) \to F(Z)$ in $\mathbb{F}$. Note that $G(f) = \mathrm{id}(Z)$ if and only if f is an edge in $\mathcal{T}$ (for then $f\alpha_A : F(Z) \xrightarrow{\sim} B$ would be in the tree and hence equal to α_B (or equivalently, would define a non-degenerate loop in the tree). Finally define for each $X \in \mathrm{Ob}(\mathbb{G})$, $\beta_X : Z = GF(X) \to X$ as that unique arrow in $\mathbb{G}$ whose image under F is $\alpha_{F(X)} : F(Z) = FGF(X) \to F(X)$, the unique path in the tree connecting $F(Z)$ and $F(X)$. By construction G is a quasi-inverse for F. We claim that $\mathbb{G}$ is free: Any arrow $X \to Y$ in G can be uniquely written (in reduced form) as a path using β_X, β_Y and non-identity images of generating edges in F since it suffices to factorize $F(f)$ in F and take the image under G of the factorization. Since $\beta_Y GF(f) \beta_X^{-1} = f$, the graph generates $\mathbb{G}$. Uniqueness follows immediately since $\alpha_B^{-1} f \alpha_A$ is the identity or is 'reduced as written'. $\square$

Since for any object A of any connected non-empty groupoid $\mathbb{F}$, the inclusion of group $\mathrm{Aut}_{\mathbb{F}}(A) \hookrightarrow \mathbb{F}$ is an essential equivalence, and any essential equivalence $F : \mathbb{G} \to \mathbb{F}$ of a *group* $\mathbb{G}$ with $\mathbb{F}$ has this form, $\mathbb{G} \xrightarrow{\sim} \mathrm{Aut}_{\mathbb{F}}(F(e)) = \mathrm{Hom}_F(F(e), F(e))$ we immediately conclude:

Corollary 2.3. *If $F : \mathbb{G} \to \mathbb{F}(\mathcal{G})$ is an essential equivalence of a group $\mathbb{G}$ with a free groupoid $\mathbb{F}(\mathcal{G})$, then $\mathbb{G}$ is a free group freely generated by the set of all paths of the form $g = \mathcal{T}_Y^{-1} f \mathcal{T}_X \neq \mathrm{id}(F(e))$ where $f \in \mathcal{G}$ and $\mathcal{T}_X$ and $\mathcal{T}_Y$ are the unique reduced paths from $F(e)$ to X and Y whose generators lie in a fixed maximal tree $\mathcal{T} \subseteq \mathcal{G}$.*

Remark. It is almost obvious to see Corollary 2.3 directly. The generators of $\mathbb{G}$ have been defined so that

$$\begin{tikzcd}
F(e) \arrow[r, "g"] \arrow[d, "t_1^{\varepsilon_1}"'] & F(e) \arrow[d, "u_1^{\varepsilon_1}"] \\
X_1 \arrow[d, "t_2^{\varepsilon_2}"'] & Y \arrow[d, "u_2^{\varepsilon_2}"] \\
\tau_X = \vdots \arrow[d, "t_i^{\varepsilon_i}"'] & \vdots = \tau_Y \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{4}$$

is commutative with $f \in \mathcal{G}$ and the t_i, u_i in the tree $\mathcal{T} \subseteq \mathcal{G}$. Thus $g = \mathrm{id}(F(e))$ if and only if $f \in \mathcal{T}$ (for then $f\mathcal{T}_X : F(e) \to Y$ would define the unique path in the tree from $F(e)$ to Y and would then be equal to $\mathcal{T}_Y$, by definition). Since for any $X \in \mathbb{G}$, its bijective correspondent, $F(X) : F(e) \to F(e)$ is in the free groupoid $\mathbb{F}(\mathcal{G})$ and thus has a unique description as a reduced path

$$F(e) \xrightarrow{f_0^{\varepsilon_0}} X_1 \xrightarrow{f_1^{\varepsilon_1}} X_2 \xrightarrow{f_2^{\varepsilon_2}} \cdots \longrightarrow X_n \xrightarrow{f_n^{\varepsilon_n}} F(e) \tag{5}$$

with the $g_i^{\varepsilon_i} \in \mathcal{G}$ $(\varepsilon = \pm 1)$ with the commutative diagram

$$\begin{tikzcd}
F(X) : F(e) \arrow[r, "f_2^{\varepsilon_0}"] & X_1 \arrow[r, "f_1^{\varepsilon_1}"] & X_2 \arrow[r] & \cdots \arrow[r] & X_n \arrow[r, "f_n^{\varepsilon_n}"] & F(e) \\
F(e) \arrow[u, "\mathrm{id}(F(e))"] \arrow[r, "g_0^{\varepsilon_0}"] & F(e) \arrow[u, dotted, "\tau_{X_1}"'] \arrow[r, "g_1^{\varepsilon_1}"] & F(e) \arrow[u, dotted, "\tau_{X_2}"'] \arrow[r] & \cdots \arrow[r] & F(e) \arrow[u, dotted, "\tau_{X_n}"'] \arrow[r, "g_n^{\varepsilon_n}"] & F(e) \arrow[u, "\mathrm{id}\, U(F(e))"']
\end{tikzcd} \tag{6}$$

giving the unique description of $F(X)$ in terms of the generators g_i.

Corollary 2.4. *Any non-empty free groupoid is equivalent to a free totally disconnected groupoid, i.e. a coproduct (=disjoint union) of free groups.*

2.5. The second proof of Theorem 2.1 will be a purely categorical (i.e. functorial) proof of Corollary 2.3.

Proof 2. Let $\mathcal{G} : E \rightrightarrows V$ be a connected non-empty graph and $\mathcal{T} : T \rightrightarrows V$ maximal tree contained in $\mathcal{G}$. We assume that $\mathcal{G}$ and $\mathcal{T}$ have degeneracies freely adjoined. Now let $\mathbb{1}$ denote the graph with exactly one vertex and exactly one (degenerate) edge and form the pushout

$$\begin{tikzcd}
\mathcal{T} \arrow[r, hook, "i"] \arrow[d, "\mathcal{T}"'] & \mathcal{G} \arrow[d, "g"] \\
\mathbb{1} \arrow[r, hook, "\bar{i}"] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V \arrow[rr, "\mathrm{id}"] \arrow[dd] & & V \arrow[dd] \\
T \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, hook] \arrow[dd] & & E \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dd] & \\
 & \mathbb{1} \arrow[rr] & & \mathbb{1} \\
\mathbb{1} \arrow[ur] \arrow[rr, hook] & & (E-T) \amalg \mathbb{1} \arrow[ur] &
\end{tikzcd} \tag{7}$$

in the category of graphs with degeneracies. Ω then has exactly one vertex and exactly one non-degenerate loop for each edge in $\mathcal{G}$ not in the tree $\mathcal{T}$. Intuitively, every edge in $\mathcal{T}$ has been collapsed to a point (i.e. made degenerate) with the edges in the complement of $\mathcal{T}$ then becoming non-degenerate loops of Ω. Note also that an edge f in $\mathcal{G}$ is sent to a degeneracy in Ω if and only if f is an edge of $\mathcal{T}$ (i.e. the square is a pullback as well as a pushout). The free groupoid functor now carries this pushout to the pushout (in **Gpd**)

$$\begin{tikzcd} \mathbb{F}(\mathscr{T}) \cong V\times V \arrow[r, hook, "\mathbb{F}(i)"] \arrow[d, "\mathbb{F}(\mathscr{T})"'] & \mathbb{F}(\mathscr{G}) \arrow[d, "\mathbb{F}(g)"] \\ \mathbb{F}(\mathbb{1}) \xrightarrow{\sim} \mathbb{1} \arrow[r, hook, "\mathbb{F}(\tilde{i})"] & \mathbb{F}(\Omega). \end{tikzcd} \tag{8}$$

Since V is non-empty, the functor $\mathbb{F}(\mathscr{T})$ is trivially an essential equivalence. Thus $\mathbb{F}(g)$ is an essential equivalence since in the category of groupoids the pushout of an essential equivalence along a monomorphism (e.g. $\mathbb{F}(i)$) is an essential equivalence. Now $\mathbb{F}(g)$ is an essential equivalence if and only if the square

$$\begin{tikzcd} \mathbb{F}(\mathscr{G}) \arrow[r, "F(g)"] \arrow[d, "\langle T, S\rangle"'] & \mathbb{F}(\Omega) \arrow[d] \\ V\times V \arrow[r] & \mathbb{1}(\xrightarrow{\sim} \mathbb{1}\times\mathbb{1}) \end{tikzcd} \tag{9}$$

is a pullback in **Gpd**. Thus the choice of any object υ in V defines a functorial section $\tilde{\upsilon}:\mathbb{1}\to V\times V$ of $V\times V\to\mathbb{1}$ and hence a functorial section $\upsilon^*:\mathbb{F}(\Omega)\to\mathbb{F}(\mathscr{G})$ of $F(g)$ by pullback. Since the new square with $\tilde{\upsilon}$ and υ^* is a pullback, the free group $\mathbb{F}(\Omega)$ is isomorphic to the group $\mathrm{Hom}_{\mathbb{F}(\mathscr{G})}(\upsilon,\upsilon)$ for any object υ in $\mathbb{F}(\mathscr{G})$. $\square$

3. Rank of any equivalent free group

3.0. The cardinality of the set of non-degenerate edges in the set of free generators $\mathbb{G}\to\mathbb{F}$ is $\mathrm{Card}(E-T)$ always. If $\mathscr{T}$ is a finite tree, then the cardinality of its set of edges is one less than the cardinality of its set of vertices. But since $\mathscr{T}$ is a maximal tree, its set of vertices is the same as the set of vertices of $\mathscr{G}$ (since $\mathscr{G}$ is connected). Thus for any equivalent group $\mathbb{G}$, $\mathrm{Rank}(\mathbb{G})=\mathrm{Card}(E-T)=\mathrm{Card}(E)-\mathrm{Card}(T)=\mathrm{Card}(E)-(\mathrm{Card}(V)-1)=\mathrm{Card}(E)-\mathrm{Card}(V)+1$, or

$$\mathrm{Rank}(\mathbb{G}) = 1-\chi(\mathscr{G}), \tag{10}$$

where $\chi(\mathscr{G})\overset{\text{def}}{=}\mathrm{Card}(V)-\mathrm{Card}(E)$ is the *Euler–Poincaré characteristic* of generating connected graph $\mathscr{G}:E\rightrightarrows V$.

4. The Nielsen–Schreier theorem

4.0. Recall that if $\mathbb{G}$ is a subgroup of a free group $\mathbb{F}(\mathscr{G})$, a *Schreier-transversal* for $\mathbb{G}$ is a set S of representatives of the set $\mathbb{F}/\mathbb{G}$ of (left) cosets of $\mathbb{F}$ modulo $\mathbb{G}$ which has the property that $e\in S$, and if $s=g_n^{\hat{\varepsilon}_n}g_{n-1}^{\hat{\varepsilon}_{n-1}}\cdots g_2^{\hat{\varepsilon}_2}g_1^{\hat{\varepsilon}_1}$ is in reduced form, then $s'=g_{n-1}^{\hat{\varepsilon}_{n-1}}g_{n-2}^{\hat{\varepsilon}_{n-2}}\cdots g_1^{\hat{\varepsilon}_1}\in S$.

Theorem 4.1 (Nielsen–Schreier [4,5]). *Any subgroup $\mathbb{G}$ of a free group $\mathbb{F}(\mathscr{I})$ is free with free generators the set of elements $t^{-1}gs \neq e$ where $s, t \in S$, a Schreier transversal, $s \in s\mathbb{G}$, $g \in \mathscr{I}$, $t \in gs\mathbb{G}$.*

Proof. Whether a group $\mathbb{F}$ is free or not, any (left, say) $\mathbb{F}$-set H defines a groupoid whose set of objects is the set H and whose set of arrows is the set $\mathbb{F} \times H$, with projection onto H as source, result of the action as target, and multiplication in $\mathbb{F}$ as composition:

$$\begin{tikzcd} \mathbb{F}\times H \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\mathrm{pr}"] \\ H \end{tikzcd} \qquad (f,x): x \longrightarrow f\cdot x \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "gf"'] & f\cdot x \arrow[d, "g"] \\ & g\cdot(f\cdot x) = gf\cdot x. \end{tikzcd} \tag{11}$$

In such a groupoid, the *set H/F of orbits* under the action is the set of connected components of the groupoid and the *subgroup of automorphisms* of any object $x \in H$ is isomorphic to the subgroup $\mathbb{F}_x = \{f \in \mathbb{F} \mid f\cdot x = x\}$ which is the *stabilizer* of the element x. Any connected component is thus a homogeneous $\mathbb{F}$-set and as a connected non-empty (if $H \neq \emptyset$) groupoid is equivalent to the group of automorphisms of any one of its objects, i.e. to the stabilizer $\mathbb{F}_x$ subgroup of any element. Any two such groups are isomorphic in the groupoid and hence conjugate as subgroups of $\mathbb{F}$. Quite trivially, if $\mathbb{F}$ is free on a set of generators $\mathscr{I}$, then any $\mathbb{F}$-set H is a free groupoid on the graph

$$\mathscr{I} \times H \overset{\alpha}{\underset{\mathrm{pr}}{\rightrightarrows}} H,$$

whose target function is the restriction of action to $\mathscr{I}$.

For our purposes here, the F-set of interest is the homogeneous $\mathbb{F}$-set $\mathbb{F}/\mathbb{G}$ of left cosets of a subgroup $\mathbb{G}$ of $\mathbb{F}$ under the usual action

$$(f, xG) \longmapsto fxG. \tag{12}$$

The resulting groupoid has as objects the left cosets and any arrow has the form dictated by the action

$$(f, xG): x\mathbb{G} \xrightarrow{f} fx\mathbb{G}. \tag{13}$$

As a groupoid, it is connected and non-empty via the coset $\mathbb{G}$ itself and the canonical functor (!) $i_{\mathbb{G}}: \mathbb{G} \hookrightarrow \mathbb{F}/\mathbb{G}$ defined by

$$g: e \longrightarrow e \longmapsto (g, \mathbb{G}): \mathbb{G} \xrightarrow{g} \mathbb{G} \tag{14}$$

is an essential equivalence. Now if $\mathbb{F}$ is a free group with a set $\mathscr{I} \subseteq \mathbb{F}$ for generators, then, by definition of the action, the groupoid $\mathbb{F}/\mathbb{G}$ is a free groupoid whose generating graph has for vertices the set $\mathbb{F}/\mathbb{G}$ of cosets and for edges the set $E = \mathscr{I} \times \mathbb{F}/\mathbb{G}$, i.e. those arrows of the form

$$\mathbb{G} \xrightarrow{g} gG, \quad g \in \mathscr{I}. \tag{15}$$

A maximal tree in this connected graph now defines a Schreier transversal since the unique path in the tree from the coset $\mathbb{G}$ to any coset $x\mathbb{G}$ defines a word in $x\mathbb{G}$, and thus a representative of the coset $x\mathbb{G}$, each of whose final segments must also be representatives in the tree. Thus Schreier's theorem is just Corollary 2.3 applied to this case. □

$$\begin{tikzcd}
 & \mathbb{G} \arrow[d] \arrow[r] & \mathbb{G} \arrow[d, "h_1^{\varepsilon_1}"] & \\
 & g_1^{\varepsilon_1}\mathbb{G} \arrow[d, "g_2^{\varepsilon}"] & & \\
s = \tau_{s\mathbb{G}} = & & h_1^{\varepsilon_1}\mathbb{G} \arrow[d] & = \tau_{gs\mathbb{G}} = t \\
 & g_2^{\varepsilon_2}g_1^{\varepsilon_1}\mathbb{G} \arrow[d, dotted] & \vdots \arrow[d] & \\
 & s\mathbb{G} \arrow[r, "g"] & gs\mathbb{G} &
\end{tikzcd} \tag{16}$$

Corollary 4.2. *If the subgroup* $\mathbb{G} \hookrightarrow \mathbb{F}(\mathcal{T})$ *has index* $i = \mathrm{Card}(V)$ *and the free group* $\mathbb{F}$ *has rank* $r = \mathrm{Card}(\mathcal{T})$, *then* $\mathrm{Card}(E) = ri$, *and from* 3.0 *we have*

$$\mathrm{Rank}(\mathbb{G}) = ri - i + 1. \tag{17}$$

5. Serre's theorem

5.0. As a second application of the main theorem, or more properly again Corollary 2.3, we will give a rather categorical proof of a theorem of Serre [6] which characterizes free groups as those which can be made to act freely on a tree.

In order to do this, we must first recall the particularization of the construction which corresponds to that of "the profunctor from $\mathbb{G}$ to $\mathbb{F}$ defined by a functor $V: \mathbb{G}^{\mathrm{op}} \times \mathbb{F} \to \mathbf{ENS}$" [1,8] to our case of interest where $\mathbb{G}$ is a group and $\mathbb{F}$ is a groupoid. It has interest outside of our context here since it is available even when choice fails. Diagramatically, it consists of the following system:

$$\begin{tikzcd}
\mathbb{G}_1 \times \mathbb{V} \underset{S}{\times} \mathbb{F}_1 \arrow[r, shift left, "\alpha_G \mathbb{F}_1"] \arrow[r, shift right, "\mathrm{pr}_{vx}^{F_1}"'] \arrow[d, shift right, "\mathbb{G} \times \alpha_F"'] \arrow[d, shift left, "\mathrm{pr}_{G \times V}"] & \mathbb{V} \underset{S}{\times} \mathbb{F}_1 \arrow[r, "\mathrm{pr}"] \arrow[d, shift right, "\alpha_F"'] \arrow[d, shift left, "\mathrm{pr}_v"] & \mathbb{F}_1 \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] \\
\mathbb{G}_1 \times \mathbb{V} \arrow[r, shift left, "\alpha G"] \arrow[r, shift right, "\mathrm{pr}_v"'] \arrow[d, "\mathrm{pr}"'] & \mathbb{V} \arrow[r, "\mathcal{T}"'] \arrow[d, "\sigma"] & \mathbb{F}_0 \\
\mathbb{G}_1 \arrow[r] & \mathbb{1} &
\end{tikzcd} \tag{18}$$

in which we have a right $(=T)$ action of the group $\mathbb{G}$ on a set $\mathbb{V}$,

$$e \xrightarrow{g} e \xrightarrow{x} \tau(x) \longmapsto x * g : e \longrightarrow \tau(x) \tag{19}$$

and a left ($=S$) action of the groupoid $\mathbb{F}$ on $\mathbb{V}$,

$$e \xrightarrow{x} \tau(x) \xrightarrow{f} T(f) \longmapsto f*x : e \longrightarrow T(f) \tag{20}$$

in which, in addition to the usual conditions for actions, 'associativity' holds, i.e. for all triplets $e \xrightarrow{g} e \xrightarrow{x} x \xrightarrow{f} y$ in $\mathbb{G}_1 \times \mathbb{V} \times_S \mathbb{F}$, we have the equality

$$f*(x*g) = (f*x)*g. \tag{21}$$

Every such system defines (via $(e,X) \mapsto (\sigma\times\tau)^{-1}(e,X) = V(e,X)$ and the actions) a functor from $\mathbb{G}^{\mathrm{op}} \times \mathbb{F}$ into sets and, conversely, every such functor defines (via $\mathbb{V} = \coprod_{(e,X)\in \mathrm{Ob}(\mathbb{G}^{\mathrm{op}}\times\mathbb{F})} V(e,X)$) a system of this type (whose quasi-diagonal $\mathbb{G}_1 \times \mathbb{V} \times \mathbb{F}_1 \rightrightarrows \mathbb{V}$ is the 'total space of the discrete fibration associated with such a functor'). Our interest here is the following easily verified

5.1. Proposition. (a) *In order that* $\mathbb{V}$ *represent a functor* $F : \mathbb{G} \to \mathbb{F}$, *i.e.* $V(e,x) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}}(F(e), X)$, *it is necessary and sufficient that* $\mathbb{V}$ *be a* torsor (*=principal homogeneous space*) under the action of the groupoid $\mathbb{F}$, *i.e.* $\mathbb{V} \neq \emptyset$ *and the canonical map* $\langle \alpha_F, \mathrm{pr}_V \rangle : \mathbb{V} \times_S \mathbb{F}_1 \to \mathbb{V} \times \mathbb{V}$ *is a bijection.*

(b) *In order that* $\mathbb{V}$ *represent a fully faithful functor* $F : \mathbb{G} \to \mathbb{F}$ *it is necessary and sufficient that* $\mathbb{V}$ *be a torsor under the action* $\mathbb{F}$ *and a pseudo-torsor under the action of* $\mathbb{G}$ *above* $\mathbb{F}_0$, *i.e.* (a) *holds and the canonical map* $\langle \mathrm{pr}_v, \alpha_G \rangle : \mathbb{G}_1 \times \mathbb{V} \to \mathbb{V} \times_{F_0} \mathbb{V}$ *is a bijection.*

(c) *In order that* $\mathbb{V}$ *represent an essential equivalence, it is necessary and sufficient that* (a) *and* (b) *hold and that, in addition,* τ *be surjective* (*i.e.* $\mathbb{V}$ *is a torsor under both actions*).

In effect, consider the groupoid $\bar{W}(\mathbb{V})$ whose set of objects is $\mathbb{V}$ and whose set of arrows is the fiber product $(\mathbb{G}_1 \times \mathbb{V})_{\alpha_G} \times_{\alpha_F} (\mathbb{V} \times_S \mathbb{F}_1)$, i.e. those 'commutative' squares of the form

$$\begin{tikzcd} e \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & x \arrow[d, "f"] \\ e \arrow[r, "v"'] & y \end{tikzcd} \tag{22}$$

where $v*g = f*u$. Then, if v_0 is a chosen point of $\mathbb{V}$, define $F(e) = \tau(v_0)$ and $F(g)$ as that unique arrow of $\mathbb{F}_1$ such that $v_0 * g = F(g) * v_0$, i.e. so that the square

$$\begin{tikzcd} e \arrow[r, "v_0"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & F(e) \arrow[d, "F(g)"] \\ e \arrow[r, "v_0"'] & F(e) \end{tikzcd}$$

is commutative. The proposition then follows immediately.

5.2. We now consider the case where we have a group $\mathbb{G}$ acting (on the right) on a non-empty graph $E \rightrightarrows V$; this means that we have actions of $\mathbb{G}$ on V and E for which the source and target maps are morphisms of $\mathbb{G}$-sets (i.e. a graph object in $\mathbb{G}$-sets). We further suppose that the action of $\mathbb{G}$ on V (and from this it follows) and on E is *free* (i.e. the canonical map $\langle \mathrm{pr}_V, \alpha_G \rangle : \mathbb{G}_1 \times V \to V \times V$ is injective), so that given any vertices v and w of V, there exists at most one $g \in \mathbb{G}$ connecting v and w: $(v=)\, w * g \xrightarrow{g} w)$. It then follows that $\mathbb{G}_1 \times V \rightrightarrows V$ and $\mathbb{G}_1 \times E \rightrightarrows E$ are isomorphic to the graphs of equivalence relations on V and E and that the orbit graph $\mathbb{G}\backslash E \rightrightarrows \mathbb{G}\backslash V$ is defined:

$$\begin{tikzcd}
\mathbb{G}_1 \times E \arrow[r, shift left, "\alpha_G"] \arrow[r, shift right, "\mathrm{pr}_E"'] \arrow[d, shift right, "G_1 \times T"'] \arrow[d, shift left, "G_1 \times S"] & E \arrow[r, "p_\varepsilon"] \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] & \mathbb{G}_1\backslash E \arrow[d, shift right, "\bar{T}"'] \arrow[d, shift left, "\bar{S}"] \\
\mathbb{G}_1 \times V \arrow[r, shift left, "\alpha_V"] \arrow[r, shift right, "\mathrm{pr}_V"'] \arrow[d, "\mathrm{pr}_G"'] & V \arrow[r] \arrow[d] & \mathbb{G}_1\backslash V \\
\mathbb{G}_1 \arrow[r] & \mathbb{1} &
\end{tikzcd} \tag{23}$$

Since the square $(\mathbb{G}_1 \times S, \mathrm{pr}, \mathrm{pr}_E, S)$ is a pullback and $(\mathrm{pr}_E, \alpha_G)$ and $(\mathrm{pr}_V, \alpha_G)$ are equivalence relations, it follows that the square

$$\begin{tikzcd}
E \arrow[r, "P_E"] \arrow[d, "S"'] & \mathbb{G}_1\backslash E \arrow[d, "\bar{S}"] \\
V \arrow[r, "P_V"'] & \mathbb{G}_1\backslash V
\end{tikzcd} \tag{24}$$

is a pullback as well. Thus we may replace (23) with

$$\begin{tikzcd}
\mathbb{G}_1 \times V \times \bar{E} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\mathrm{pr}"'] \arrow[d, shift right, "G \times \alpha"'] \arrow[d, shift left, "G \times \mathrm{pr}"] & V \underset{v}{\times} \bar{E} \arrow[r, "\mathrm{pr}"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\mathrm{pr}"] & \bar{E}\,(=\mathbb{G}_1\backslash E) \arrow[d, shift right, "\bar{T}"'] \arrow[d, shift left, "\bar{S}"] \\
\mathbb{G}_1 \times V \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\mathrm{pr}_V"'] \arrow[d] & V \arrow[r, "P_V"'] \arrow[d] & V\,(=\mathbb{G}_1\backslash V) \\
\mathbb{G}_1 \arrow[r] & \mathbb{1} &
\end{tikzcd} \tag{25}$$

and view the graph $V \times_S \bar{E} \rightrightarrows V$ as having vertices of the form

$$e \xrightarrow{\ v\ } P_V(v) \tag{26}$$

and edges of the form

$$
\begin{tikzcd}
e \arrow[r, "\upsilon"] \arrow[dr, "\alpha * \upsilon"'] \arrow[r, phantom, shift right=3ex, "\swarrow (\upsilon,\alpha)"] & P_V(\upsilon) \arrow[d, "\alpha"] \\
& P_V(\alpha * \upsilon).
\end{tikzcd}
\tag{27}
$$

where $\alpha * \upsilon$ is the map induced by $T: E \to V$ and the free action of $\mathbb{G}$ is given by

$$
\begin{tikzcd}
e \arrow[d, "g"'] \arrow[drr, "\upsilon * g"] \arrow[ddrr] \arrow[drr, phantom, shift right=3ex, "\swarrow (\upsilon,-)*g"] & & \\
e \arrow[rr, "\upsilon"'] \arrow[drr, "\alpha * \upsilon"'] \arrow[drr, phantom, shift left=2ex, "\swarrow \alpha * y * g"] & & P_V(\upsilon) \arrow[d, "\alpha"] \\
& & P_V(\alpha * \upsilon)
\end{tikzcd}
\tag{28}
$$

This forces the target vertex $\alpha * \upsilon$ to be unique since given any edge of the form

$$
\begin{tikzcd}
e \arrow[dr, "\upsilon"] \arrow[ddr, "w"'] \arrow[dr, phantom, shift right=3ex, "\swarrow"] & \\
& P_V(\upsilon) \arrow[d, "\alpha"] \\
& P_V(\alpha * \upsilon)
\end{tikzcd}
\tag{29}
$$

there exists a unique $g \in \mathbb{G}_1$ such that $(\alpha * \upsilon) * g = w$ and $\upsilon * g = \upsilon$. But then $g = e$ and hence $\alpha * \upsilon = w$. From this it now follows that $\mathbb{G}$ operates freely on the free groupoid $F_1(V \times \bar{E}) \rightrightarrows V$ generated by $V \times \bar{E} \rightrightarrows V$ and that $\mathbb{G}_1 \times_S F_1(V \times \bar{E}) \rightrightarrows \mathbb{G}_1 \times V$ is the free groupoid generated by $\mathbb{G} \times V \times \bar{E} \rightrightarrows \mathbb{G}_1 \times V$. Consequently, the free groupoid $\mathbb{F}_1(\bar{E}) \rightrightarrows V$ is the quotient groupoid under this action and the square

$$
\begin{tikzcd}
\mathbb{F}_1(V \times_S \bar{E}) \arrow[r, "\mathbb{F}_1(\mathrm{pr})"] \arrow[d, "S"] & \mathbb{F}_1(\bar{E}) \arrow[d, "S"] \\
V \arrow[r, "P_\upsilon"'] & \bar{V}
\end{tikzcd}
\tag{30}
$$

is a pullback so that we have a system

$$
\begin{tikzcd}
\mathbb{G}_1 \times V \times F_1(\bar{E}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & V \times \mathbb{F}_1(\bar{E}) \arrow[r, "\mathrm{pr}"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\mathrm{pr}"] & \mathbb{F}_1(\bar{E}) \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] \\
\mathbb{G}_1 \times V \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & V \arrow[r] \arrow[d] & \bar{V} \\
\mathbb{G}_1 \arrow[r] & \mathbb{1} &
\end{tikzcd}
\tag{31}
$$

exactly as in (18) whose rows are torsors as in (b) of Proposition 5.1. But the system is then equivalent to a representation of an essential equivalence $F: \mathbb{G} \to \mathbb{F}_1(\bar{E})$ if and only if the middle column of (31) is a torsor, i.e. if and only if the canonical map $F_1 \times F_1 : (V \times \bar{E}) \to V \times V$ is a bijection and thus if and only if $V \times \bar{E} \rightrightarrows V$ was a tree. Since the target of F is the free groupoid $F_1(\bar{E})$, it then follows that $\mathbb{G}$ is a free group by Corollary 2.3 and we have the

Theorem 5.3. (Serre [6,7]). *A group $\mathbb{G}$ is free if and only if $\mathbb{G}$ can be made to act freely on a tree.*

The 'only if' is a consequence of the free action of $\mathbb{G}$ on the tree $\mathbb{G} \times S \overset{\mathrm{pr}}{\rightrightarrows} \mathbb{G}$ where $S \subseteq \mathbb{G}$ is a set of free generators for $\mathbb{G}$; the orbit graph is then $S \to 1$.

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